Technical Notes

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Automated Treatment of Problems in Stochastic Processes

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THIS paper is concerned with continuous stochastic processes expressible in the form

$$X(t)=\sum_{n=1}^{N}\left(\frac{S_n}{M\pi}\right)^{1/2}\sum_{m=1}^{M}\sin(\omega_n t+\psi_{mn}) \tag{1}$$

where the ω_n's and S_n's are given constants, M is a large number, and the ψ_{mn}'s are $M\times N$ statistically independent, uniformly distributed random variables on the interval $[0,2\pi]$. X then is a random variable for any value of t and thus $X(t)$ is a random process or, in different terminology, a random function of t. The function is clearly continuous and differentiable for all t.

In order to examine the statistical nature of $X(t)$, the mean and autocorrelation functions for the process must be found. Taking the expectation of both sides of Eq. (1), it is seen that

$$E[X(t)]=0 \tag{2}$$

The autocorrelation function is

$$R(\tau)\equiv E[x(t)x(t+\tau)]=\sum_{n=1}^{N}\frac{S_n}{2\pi}\cos\omega_n\tau \tag{3}$$

Thus, the process is at least weakly stationary. It will be shown that the process is asymptotically Gaussian (with respect to M) and, therefore, strictly stationary.

The power spectral density $\Phi(\omega)$ is found to be

$$\Phi(\omega)=\sum_{n=1}^{N}S_n\frac{[\delta(\omega+\omega_n)+\delta(\omega-\omega_n)]}{2} \tag{4}$$

which is a symmetrical distribution of discrete frequencies.

One convenient way to demonstrate the Gaussian nature of the limit process is through the use of the characteristic functional. The characteristic functional, $M_z[\theta(t)]$, of a process, $Z(t)$, defined on an interval T is defined by[1]:

$$M_z[\theta(t)]=E\left\{\exp\left[i\int_T\theta(t)Z(t)\,dt\right]\right\} \tag{5}$$

A Gaussian process, $Z(t)$, may be defined as one whose characteristic functional is of the form[1]

$$M_z[\theta(t)]=\exp\left[i\int_T\mu_z(t)\theta(t)\,dt-\tfrac{1}{2}\int_T\int K_{zz}(t_1,t_2)\theta(t_1)\theta(t_2)\,dt_1\,dt_2\right] \tag{6}$$

where $\mu_z(t)$ and $K_{zz}(t_1,t_2)$ are the mean and covariance functions of the process.

Rayleigh[2] showed that if a large number of sinusoids of the same frequency with phases randomly distributed be superposed, then the resulting sinusoid has an amplitude that is a Rayleigh distributed random variable. That is,

$$\lim_{M\to\infty}\frac{1}{(M)^{1/2}}\sum_{m=1}^{M}\sin(\omega t+\psi_m)=A\sin(\omega t+\psi) \tag{7}$$

where A is a Rayleigh distributed random variable with variance 1/2, and ψ is a uniformly distributed random variable on the interval $[0,2\pi]$ statistically independent of A. Thus if M is large, each term of the sum on n in Eq. (1) can be written as

$$X_n(t)=B_n\sin(\omega_n t+\psi_n) \tag{8}$$

where B_n is Rayleigh distributed with parameter $S_n/2\pi$, and ψ_n is uniformly distributed and independent of B_n. That is, the density functions for B_n and ψ_n are:

$$p(b_n)=(2\pi b_n/S_n)\exp[-\pi b_n^2/S_n] \qquad (0<b_n) \tag{9}$$

$$p(\psi_n)=1/2\pi \qquad 0<\psi_n<2\pi \tag{10}$$
$$0 \qquad \text{otherwise}$$

The characteristic functional for the process, $X_n(t)$, is

$$M_{x_n}[\theta(t)]=\int_0^\infty\int_0^{2\pi}\exp\left[i\int_{-\infty}^{\infty}\theta(t)b_n\sin(\omega_n t+\psi_n)\,dt\right]\times p(\psi_n)p(b_n)\,d\psi_n\,db_n=\frac{1}{2\pi}\int_0^\infty p(b_n)\,db_n\int_0^{2\pi}\times\exp[ib_n(a\cos\psi_n+c\sin\psi_n)]\,d\psi_n \tag{11}$$

where

$$a=\int_{-\infty}^{\infty}\theta(t)\sin\omega_n t\,dt$$
$$c=\int_{-\infty}^{\infty}\theta(t)\cos\omega_n(t)\,dt \tag{12}$$

which can be written in the form

$$M_{x_n}[\theta(t)]=\frac{1}{2\pi}\int_0^\infty p(b_n)\,db_n\int_0^{2\pi}\exp[iz\sin(\psi_n+\phi)]\,d\psi_n \tag{13}$$

where

$$z=b_n(a^2+c^2)^{1/2}; \qquad \phi=\arctan[a/c] \tag{14}$$

The inner integral can be evaluated[3] as

$$\frac{1}{2\pi}\int_0^{2\pi}\exp[iz\sin(\psi_n+\phi)]\,d\psi_n=J_0(z) \tag{15}$$

where J_0 is the zero order Bessel function of the first kind.

The characteristic functional assumes the form

$$M_{x_n}[\theta(t)]=\int_0^\infty J_0[b_n(a^2+c^2)^{1/2}]\frac{2\pi b_n}{S_n}\exp\left[-\frac{\pi b_n^2}{S_n}\right]db_n \tag{16}$$

which can be integrated[3] as

$$M_{x_n}[\theta(t)]=\exp\left(-\frac{(a^2+c^2)S_n}{4\pi}\right)$$

or in more explicit form

$$M_{x_n}[\theta(t)]=\exp\left\{-\frac{S_n}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta(t_1)\theta(t_2)\cos\omega_n(t_2-t_1)\,dt_1\,dt_2\right\}$$

from Eq. (3) we see that this is

$$M_{x_n}[\theta(t)]=\exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta(t_1)\theta(t_2)R_n(t_2-t_1)\,dt_1\,dt_2\right\} \tag{17}$$

but since the mean of $X_n(t)$ is zero $R_n(\cdot)$ is the covariance function, $K_{x_nx_n}(t_2-t_1)$, and thus Eq. (17) is of the form Eq. (6), and therefore, the process $X_n(t)$ is Gaussian (or more accurately it is approximately Gaussian for large finite M). Since the sum of independent Gaussian processes is Gaussian, it follows that the process defined in Eq. (1) is a Gaussian process having zero mean and correlation function given by Eq. (3).

If a given stationary process is known to be Gaussian with zero mean and known power spectral density function, it can be

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approximately represented by Eq. (1). Two approximations are involved: one of these, the finiteness of M, has already been mentioned; the other is that of replacing a continuous power spectrum by a discrete one (if, of course, the given process has a continuous power spectrum). It is clear that the larger we choose M, the more nearly does the representation, Eq. (1), approximate a normal process. The replacement of the continuous spectrum by a discrete one has a more subtle effect on the validity of the representation. A Gaussian process with a continuous spectral density is known to be ergodic. It can be shown that the process Eq. (1) is nonergodic. This is easily done by considering the criterion that a stationary Gaussian process, $X(t)$, be ergodic[1,4]

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |K_{xx}(\tau)|^2\,d\tau = 0 \tag{18}$$

This condition is not satisfied for a Gaussian process with a discrete power spectral density.

One may then ask, "How can Eq. (1) be used as a representation of an ergodic Gaussian process?" The answer is that so long as one uses ensemble statistics (as opposed to sample statistics), the lack of ergodicity of Eq. (1) is irrelevant provided that the discrete frequencies are chosen close enough together and the time intervals involved are not overly long.

Almost any statistical question that may be asked about a stationary Gaussian process of given power spectral density may be answered to any degree of accuracy required by using Eq. (1) to generate a sequence of time histories and then taking the ratio of the number of records having the required property to the total number of records. For specific problems the number of records required to get meaningful statistical data may be large or small depending on whether the events of interest have a low or high probability of occurrence. Typically, if one studies an event having a probability of occurrence of $\frac{1}{4}$, then perhaps 40 or 50 time histories are sufficient to determine this probability. If, on the other hand, one is concerned with low-probability events (such as first passage times of high crossing levels), then perhaps five or ten thousand records may be required. The question of the feasibility of such an approach is immediately suggested.

Modern, high-speed, digital computers are ideally suited to the routine procedure of generating records from Eq. (1). Standard techniques exist for generating large numbers of independent random numbers from a uniformly distributed population in a very short time (of the order of a few seconds to generate 100,000 numbers). Using such a subprogram it is very simple to write a program that will evaluate Eq. (1) at any specified time intervals and thus produce a digital record of a sample from the given random process. The program may be written to sequentially generate as many records as may be deemed necessary. The analysis of these records for the occurrence or nonoccurrence of events of specified type can be done internally on the computer and a printout of the probability of these events is readily obtained.

In order to test the feasibility of this procedure, a program was written to determine the probability distribution of first passage times for a wide band Gaussian process. The levels studied were fairly low (1.5–2σ). A printout of the relative frequency of occurrence (based on analysis of 500 records) of first passage before time T for 100 values of T was obtained in about 7 min on a CDC 6600 computer. Of course, first passage distributions for higher levels take longer, but if one is faced with the necessity of getting a solution it may well be that computer times of the order of several hours is tolerable. Thus, from an engineering point of view, the first passage problem (as well as many other statistical problems) may be regarded as soluble.

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Comparison of Numerical and Asymptotic Expansion Solutions of the Boundary-Layer Equations

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RECENTLY several authors have presented asymptotic solutions of the boundary-layer equations for limiting cases. In order to assist in establishing the validity of these procedures numerical calculations corresponding to two of these solutions have been carried out. The first considered is that of Brown and Stewartson[1] for the reverse flow solutions of the Falkner-Skan equation when $\beta \to 0^-$ and $f''(0) \to 0^-$. The second is that of Kassoy[2] for the laminar unit Prandtl number compressible boundary layer with blowing when $\beta \to 0^+$ and $f''(0) \to 0^+$.

Brown and Stewartson obtained an asymptotic solution of the Falkner-Skan equation†

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0 \tag{2a, b}$$

$$f'(\eta \to \infty) \to 1 \tag{2c}$$

for reverse flows ($f''(0) < 0$) in the limit as $\beta \to 0^-$. Here $f(\eta)$ is a nondimensional stream function and the primes denote differentiation with respect to the independent similarity variable η. They showed that in the limit as $\beta \to 0^-$ that the nondimensional shearing stress at the surface is

$$f''(0) = -C(-\beta)^{3/4} \tag{3}$$

where the constant $C = 1.544$ is obtained by numerical integration of an auxiliary equation.

In the present work numerical solutions of Eq. (1) subject to the boundary conditions given in Eq. (2) have been obtained in the limit as $\beta \to 0^-$ and $f''(0) \to 0^-$. The details of the numerical integration scheme are given in Ref. 3. Comparison of the numerical results for $f''(0)$ and the asymptotic solution of Brown and Stewartson[1] is shown in Table 1 and Fig. 1. The relative error given in Table 1 is defined as

$$\text{Relative error} = \frac{f''(0)_{\text{analytical}} - f''(0)_{\text{numerical}}}{f''(0)_{\text{numerical}}}$$

In the limit as $\beta \to 0^-$ these results show excellent agreement between the asymptotic solution of Brown and Stewartson and the present numerical results.

Examination of the nondimensional velocity profiles shows that they exhibit the characteristic reverse flow profile. As $\beta \to 0^-$ the magnitude of the negative velocity decreases while the extent of the reverse flow region increases. Further, small values of shearing stress near the surface for $\beta \to 0^-$ indicate that in this region the effects of the small adverse pressure gradient dominate. However, away from the surface the shearing stress increases signifi-

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† The notation is that of Ref. 1.